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1999 J. Phys. A: Math. Gen. 32 L143

(http://iopscience.iop.org/0305-4470/32/13/001)

J. Phys. A: Math. Gen. **32** (1999) L143–L150. Printed in the UK PII: S0305-4470(99)00141-9

**LETTER TO THE EDITOR**

# Partially reversible operation and information gain in quantum measurement processes

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Received 14 December 1998

**Abstract.** A partially reversible operation is introduced in quantum measurement processes and the information-theoretical property is investigated. The dual map of a partially reversible operation becomes reversible with respect to the intrinsic observable of a physical system. It is shown that the amount of information about the intrinsic observable of the physical system, obtained in a quantum measurement process described by a partially reversible operation, is equal to the decrease of the entropy of the measured physical system.

The state change of a physical system that is caused by a quantum measurement process is described by a completely positive map, called a quantum operation [1–6]. Such a state change cannot be described, in general, by a unitary transformation. If the state change is given by some unitary transformation, the initial state can be recovered by the inverse of the unitary transformation. In general, it is impossible to recover the initial quantum state when the state change is described by arbitrary quantum operations. Ueda and Kitagawa, however, found the logical reversibility in the continuous measurement of photon number with the quantum counter [7], and Imamoḡlu found it in the quantum nondemolition measurement of photon number [8]. Royer obtained the quantum measurement process of a spin-$\frac{1}{2}$ system, where the initial quantum state can be recovered from the post-measurement state with finite probability [9]. Furthermore, Mabuchi and Zoller showed that the reverse of the state change caused by the quantum measurement process is possible under the appropriate conditions [10]. Recently Nielsen *et al* have found the conditions for a quantum operation to describe the reversible state change of a physical system [5, 6] and they have shown that such a reversible operation is closely related to the quantum teleportation [5] and to the quantum error correcting code [6]. A quantum measurement process which can be described by a reversible operation gives us no information about the quantum state of the measured physical system. In this letter, we introduce a partially reversible operation in a quantum measurement process, and we obtain the condition for a quantum measurement process to be described by a partially reversible operation. It is shown that in a quantum measurement process described by a partially reversible operation, the information gain about the intrinsic observable of the physical system is equal to the entropy decrease of the measured physical system.

We briefly summarize the basic formulation of quantum measurement processes. To measure some intrinsic observable of a physical system, we first prepare an appropriate measurement apparatus and then we make the interaction between the measurement apparatus and the physical system to create some quantum correlation between them. After the interaction, we obtain the value exhibited by the measurement apparatus. Let $\hat{\mathcal{X}}_S(x)$ be a positive operator-valued measure (POVM) which represents the intrinsic observable and $\hat{\mathcal{Y}}_A(y)$ be a POVM of the measurement apparatus that describes the readout process of the measurement outcome $y$. For the sake of simplicity, we assume that the parameters $x$ and $y$ take only discrete values. We denote the spectral set of the intrinsic observable as $\Omega_X$ and the set of all possible measurement outcomes as $\Omega_Y$. In this case, we have $\sum_{x\in\Omega_X}\hat{\mathcal{X}}_S(x)=\hat{1}_S$ and $\sum_{y\in\Omega_Y}\hat{\mathcal{Y}}_A(y)=\hat{1}_A$, where $\hat{1}_S$ and $\hat{1}_A$ are identity operators in the Hilbert spaces $\mathcal{H}_S$ and $\mathcal{H}_A$ of the physical system and the measurement apparatus. Furthermore, let $\hat{\mathcal{U}}_{SA}$ be a unitary operator which describes the time-evolution of the system–apparatus compound system. If the quantum state of the physical system before the interaction with the measurement apparatus is described by a statistical operator $\hat{\rho}_{\text{in}}^S$ and the measurement apparatus is prepared in the quantum state $\hat{\rho}_{\text{in}}^A$, the quantum state of the system–apparatus compound system just before the read-out of the measurement outcome becomes $\hat{\rho}_{\text{out}}^{SA}=\hat{\mathcal{U}}_{SA}(\hat{\rho}_{\text{in}}^S\otimes\hat{\rho}_{\text{in}}^A)\hat{\mathcal{U}}_{SA}^\dagger$. Then the output probability $P_{\text{out}}^A(y)$ of the measurement apparatus and the quantum state $\hat{\rho}_{\text{out}}^S(y)$ of the physical system after obtaining the measurement outcome $y$ are given respectively by the following formulae [1, 2]

$$P_{\text{out}}^A(y)=\text{Tr}_S[\hat{\mathcal{L}}_S(y)\hat{\rho}_{\text{in}}^S] \qquad \hat{\rho}_{\text{out}}^S(y)=\frac{\hat{\mathcal{L}}_S(y)\hat{\rho}_{\text{in}}^S}{\text{Tr}_S[\hat{\mathcal{L}}_S(y)\hat{\rho}_{\text{in}}^S]} \tag{1}$$

where $\hat{\mathcal{L}}_S(y)$ is the quantum operation of the physical system, which is defined for an arbitrary operator $\hat{O}_S$ defined on the Hilbert space $\mathcal{H}_S$,

$$\hat{\mathcal{L}}_S(y)\hat{O}_S=\text{Tr}_A[(\hat{1}_S\otimes\hat{\mathcal{Y}}_A(y))\hat{\mathcal{U}}_{SA}(\hat{O}_S\otimes\hat{\rho}_{\text{in}}^A)\hat{\mathcal{U}}_{SA}^\dagger]. \tag{2}$$

In the physical system before the interaction with the measurement apparatus, the intrinsic observable takes the value $x$ with probability $P_{\text{in}}^S(x)=\text{Tr}_S[\hat{\mathcal{X}}_S(x)\hat{\rho}_{\text{in}}^S]$, and when after obtaining the measurement outcome $y$, it takes the value $x$ with probability $P_{\text{out}}^S(x|y)=\text{Tr}[\hat{\mathcal{X}}_S(x)\hat{\rho}_{\text{out}}^S(y)]$ in the post-measurement state of the physical system. The information-theoretical properties of quantum measurement processes have been investigated in detail [11–13].

The quantum operation $\hat{\mathcal{L}}_S(y)$ given by equation (2), which is a trace-decreasing and completely positive map, can be represented by the decomposition formula [2]

$$\hat{\mathcal{L}}_S(y)\hat{O}_S=\sum_{j\in\mathcal{M}}\hat{M}_j^S(y)\hat{O}_S\hat{M}_j^{S\dagger}(y) \tag{3}$$

which is not uniquely determined by the quantum operation $\hat{\mathcal{L}}_S(y)$. For example, if we use the spectral decomposition of the initial statistical operator of the measurement apparatus, $\hat{\rho}_{\text{in}}^A=\sum_{k\in\mathcal{N}}p_k|\phi_A(k)\rangle\langle\phi_A(k)|$, we obtain the decomposition operator,

$$M_j^S(y)=\sqrt{p_k}\langle\phi_A(k')|(\hat{1}_S\otimes\hat{\mathcal{Y}}_A^{1/2}(y))\hat{\mathcal{U}}_{SA}|\phi_A(k)\rangle \tag{4}$$

where $j=(k,k')$ and $\mathcal{M}=\mathcal{N}\times\mathcal{N}$. The output probability $P_{\text{out}}^A(y)$ of the measurement apparatus is usually expressed in terms of the initial quantum state $\hat{\rho}_{\text{in}}^S$ and the POVM $\hat{\mathcal{Z}}_S(y)$, called the detection (decision) operator [14] or the operational observable [15, 16], of the physical system such that $P_{\text{out}}^A(y)=\text{Tr}_S[\hat{\mathcal{Z}}_S(y)\hat{\rho}_{\text{in}}^S]$, where the POVM $\hat{\mathcal{Z}}_S(y)$ is given by

$$\hat{\mathcal{Z}}_S(y)=\hat{\mathcal{L}}_S^\dagger(y)\hat{1}_S=\sum_{j\in\mathcal{M}}\hat{M}_j^{S\dagger}(y)\hat{M}_j^S(y) \tag{5}$$

which satisfies $\sum_{y\in\Omega_Y}\hat{\mathcal{L}}_S^\dagger(y)\hat{1}_S=\hat{1}_S$. Here the dual map $\hat{\mathcal{L}}_S^\dagger(y)$ of the quantum operation $\hat{\mathcal{L}}_S(y)$ is defined by the relation $\text{Tr}_S[\hat{O}_S\hat{\mathcal{L}}_S(y)\hat{O}_S']=\text{Tr}_S[\hat{\mathcal{L}}_S^\dagger(y)O_S\cdot\hat{O}_S']$, that is,

$$\hat{\mathcal{L}}_S^\dagger(y)\hat{O}_S=\text{Tr}_A[\hat{\mathcal{U}}_{SA}^\dagger(\hat{O}_S\otimes\hat{\mathcal{Y}}_A(y))\hat{\mathcal{U}}_{SA}(\hat{1}_S\otimes\hat{\rho}_{\text{in}}^A)]$$

$$= \sum_{j \in \mathcal{M}} \hat{M}_j^{S\dagger}(y)\hat{O}_S\hat{M}_j^S(y). \tag{6}$$

The formulae given by equations (1)–(6) that describe quantum measurement processes of discrete observables can be generalized for quantum measurement processes of continuous observables [3, 4].

The quantum operation $\hat{\mathcal{L}}_S(y)$ that yields the statistical operator $\hat{\rho}_{\mathrm{out}}^S(y)$, which is conditional on the measurement outcome $y$, from the initial statistical operator $\hat{\rho}_{\mathrm{in}}^S$ is called the reversible operation [5, 6], if there exits a reversal operation $\hat{\mathcal{R}}_S(y)$ such that

$$\hat{\mathcal{R}}_S(y)\left\{\frac{\hat{\mathcal{L}}_S(y)\hat{\rho}_{\mathrm{in}}^S}{\mathrm{Tr}_S[\hat{\mathcal{L}}_S(y)\hat{\rho}_{\mathrm{in}}^S]}\right\} = \hat{\rho}_{\mathrm{in}}^S \tag{7}$$

which is written from equation (1),

$$\hat{\mathcal{R}}_S(y)[\hat{\mathcal{L}}_S(y)\hat{\rho}_{\mathrm{in}}] = P_{\mathrm{out}}^A(y)\hat{\rho}_{\mathrm{in}} \tag{8}$$

where the support of the initial statistical operator $\hat{\rho}_{\mathrm{in}}^S$ is confined to the subspace $\mathcal{H}_S^{(0)}$ of the Hilbert space $\mathcal{H}_S$. The reversal operation $\hat{\mathcal{R}}_S(y)$ is a trace-preserving and completely positive map. The reversible operation is closely related to the quantum teleportation [5] and the quantum error correcting code [6]. It is shown that any measurement process described by a reversible quantum operation gives us no information about the initial quantum state $\hat{\rho}_{\mathrm{in}}^S$ of the measured system [5, 6], which is consistent with indistinguishability of quantum states. Thus the output probability $P_{\mathrm{out}}^A(y) = \mathrm{Tr}_S[\hat{\mathcal{L}}_S(y)\hat{\rho}_{\mathrm{in}}^S]$ for the reversible operation does not depend on the initial quantum state $\hat{\rho}_{\mathrm{in}}^S$ of the measured physical system.

We now introduce a partially reversible operation. When the intrinsic observable $\hat{\mathcal{X}}_S(x)$ that we want to measure is given by a one-dimensional projection operator such that $\hat{\mathcal{X}}_S(x) = |\psi_S(x)\rangle\langle\psi_S(x)|$ and the set $\{|\psi_S(x)\rangle \mid x \in \Omega_X\}$ is a complete orthonormal system of the Hilbert space $\mathcal{H}_S$, any operator $\hat{O}_S$ of the physical system is determined by giving all the matrix elements $\langle\psi_S(x)|\hat{O}_S|\psi_S(x')\rangle$. Then, roughly speaking, equation (7) requires that all the matrix elements $\langle\psi_S(x)|\hat{\rho}_{\mathrm{in}}^S|\psi_S(x')\rangle$ should be recovered from the matrix elements $\langle\psi_S(x)|\hat{\rho}_{\mathrm{out}}^S(y)|\psi_S(x')\rangle$. For the partially reversible operation with respect to the intrinsic observable $\hat{\mathcal{X}}_S(x)$, we require that the diagonal elements of the initial quantum state $\hat{\rho}_{\mathrm{in}}^S$ should be recovered from the diagonal elements of the quantum state $\hat{\rho}_{\mathrm{out}}^S(y)$ by some quantum operation $\hat{\mathcal{R}}_S(y)$. Here it is assumed that the dual map, $\hat{\mathcal{R}}_S^\dagger(y)$, of the reversal operation induces the transformation of the intrinsic observable such that $\hat{\mathcal{X}}_S(x) \to \hat{\mathcal{X}}_S(f_y^{-1}(x))$, where $f_y(x) \in \Omega_X$ is an invertible function of $x \in \Omega_X$, conditional on the measurement outcome $y \in \Omega_Y$. In this case, we have the quantum operation $\hat{\mathcal{R}}_S^{\dagger-1}(y)$ which satisfies the relation $\hat{\mathcal{R}}_S^{\dagger-1}(y)\hat{\mathcal{X}}_S(x) = \hat{\mathcal{X}}_S(f_y(x))$. This means that the initial probability $P_{\mathrm{in}}^S(x)$ of the intrinsic observable can be obtained from its post-measurement probability $P_{\mathrm{out}}^S(x|y)$ by changing the argument $x$ as $x \to f_y^{-1}(x)$. Then the quantum operation $\hat{\mathcal{L}}_S(y)$ becomes partially reversible with respect to the intrinsic observable $\hat{\mathcal{X}}_S(x)$ if it satisfies

$$\mathrm{Tr}_S\{\hat{\mathcal{X}}_S(x)\hat{\mathcal{R}}_S(y)[\hat{\mathcal{L}}_S(y)\hat{\rho}_{\mathrm{in}}]\} = P_{SA}(y|x)\,\mathrm{Tr}_S[\hat{\mathcal{X}}_S(x)\hat{\rho}_{\mathrm{in}}] \tag{9}$$

where $P_{SA}(y|x)$ is a function of the measurement outcome $y$, conditional on the value $x$ of the intrinsic observable $\hat{\mathcal{X}}_S(x)$ of the physical system. In this equation, it is not necessary that the intrinsic observable $\hat{\mathcal{X}}_S(x)$ is a one-dimensional projection operator. Although we cannot obtain any information about the initial quantum state of the physical system in the quantum measurement process described by the reversible operation, we can obtain some information in the quantum measurement process described by the partially reversible operation.

Let us consider the properties of the partially reversible operation $\hat{\mathcal{L}}_S(y)$. We first obtain from equation (9)

$$\begin{aligned}\mathrm{Tr}_S\{\hat{\mathcal{L}}_S^\dagger(y)[\hat{\mathcal{R}}_S^\dagger(y)\hat{\mathcal{X}}_S(x)]\cdot\hat{\rho}_{\mathrm{in}}^S\} &= \mathrm{Tr}_S\{\hat{\mathcal{L}}_S^\dagger(y)\hat{\mathcal{X}}_S(f_y^{-1}(x))\cdot\hat{\rho}_{\mathrm{in}}^S\}\\ &= P_{\mathrm{out}}^{SA}(y|x)\,\mathrm{Tr}_S[\hat{\mathcal{X}}_S(x)\hat{\rho}_{\mathrm{in}}].\end{aligned} \tag{10}$$

If this equality holds for any initial statistical operator $\hat{\rho}_{\mathrm{in}}^S$ of the physical system, the following relation must be satisfied

$$\hat{\mathcal{L}}_S^\dagger(y)\hat{\mathcal{X}}_S(x) = P_{SA}(y|f_y(x))\hat{\mathcal{X}}_S(f_y(x)) \tag{11}$$

which indicates that except for the proportional factor $P_{SA}(y|x)$, the intrinsic observable $\hat{\mathcal{X}}_S(x)$ of the physical system is covariant under the dual map $\hat{\mathcal{L}}_S^\dagger(y)$ of the partially reversible operation. Furthermore, since $\hat{\mathcal{X}}_S(f_y(x)) = \hat{\mathcal{R}}_S^{\dagger\,-1}(y)\hat{\mathcal{X}}_S(x)$, equation (11) is expressed as

$$\hat{\mathcal{R}}_S^\dagger(y)[\hat{\mathcal{L}}_S^\dagger(y)\hat{\mathcal{X}}_S(x)] = P_{SA}(y|f_y(x))\hat{\mathcal{X}}_S(x) \tag{12}$$

which means that the dual map of the partially reversible operation with respect to the intrinsic observable $\hat{\mathcal{X}}_S(x)$ is the reversible operation for the intrinsic observable $\hat{\mathcal{X}}_S(x)$. If the intrinsic observable $\hat{\mathcal{X}}_S(x)$ is a $d$-dimensional projector such that $\mathrm{Tr}_S\,\hat{\mathcal{X}}_S(x) = d$, we obtain

$$\hat{\mathcal{R}}_S^\dagger(y)\left\{\frac{\hat{\mathcal{L}}_S^\dagger(y)\hat{\mathcal{X}}_S(x)}{\mathrm{Tr}_S[\hat{\mathcal{L}}_S^\dagger(y)\hat{\mathcal{X}}_S(x)]}\right\} = \frac{1}{d}\hat{\mathcal{X}}_S(x) \tag{13}$$

which is compared with equation (7) for the reversible operation.

The output probability $P_{\mathrm{out}}^A(y)$ of the measurement apparatus is calculated from equations (1) and (11). The linearity of the quantum operation $\hat{\mathcal{L}}_S^\dagger(y)$ yields

$$\begin{aligned}P_{\mathrm{out}}^A(y) &= \mathrm{Tr}_S[\hat{\mathcal{L}}_S^\dagger(y)\hat{1}_S\cdot\hat{\rho}_{\mathrm{in}}^S]\\ &= \sum_{x\in\Omega_X}\mathrm{Tr}_S[\hat{\mathcal{L}}_S^\dagger(y)\hat{\mathcal{X}}_S(x)\cdot\hat{\rho}_{\mathrm{in}}^S]\\ &= \sum_{x\in\Omega_X}P_{SA}(y|f_y(x))\,\mathrm{Tr}_S[\hat{\mathcal{X}}_S(f_y(x))\hat{\rho}_{\mathrm{in}}^S]\\ &= \sum_{x\in\Omega_X}P_{SA}(y|f_y(x))P_{\mathrm{in}}^S(f_y(x)).\end{aligned} \tag{14}$$

If the functions $P_{SA}(y|x)$ and $f_y(x)$ and the spectral set $\Omega_X$ satisfy the relation,

$$\sum_{x\in\Omega_X}P_{SA}(y|f_y(x))F(f_y(x)) = \sum_{x\in\Omega_X}P_{SA}(y|x)F(x) \tag{15}$$

for any non-singular function $F(x)$, we obtain the relation between the probability $P_{\mathrm{in}}^S(x)$ of the intrinsic observable in the initial quantum state and the output probability $P_{\mathrm{out}}^A(y)$ of the measurement apparatus,

$$P_{\mathrm{out}}^A(y) = \sum_{x\in\Omega_X}P_{SA}(y|x)P_{\mathrm{in}}^S(x). \tag{16}$$

Thus if equation (15) holds, the function $P_{SA}(y|x)$ appearing in equation (9) is the conditional probability that the measurement outcome $y$, is obtained when the intrinsic observable of the physical system takes the value $x$ in the initial quantum state $\hat{\rho}_{\rm in}^S$. In this case, the amount of information $I(Y_{\rm out}^A; X_{\rm in}^S)$ about the intrinsic observable of the physical system, obtained in the quantum measurement process, can be represented by Shannon mutual entropy [12],

$$I(Y_{\rm out}^A; X_{\rm in}^S) = \sum_{x\in\Omega_X}\sum_{y\in\Omega_Y} P_{SA}(y|x)P_{\rm in}^S(x)\log\left[\frac{P_{SA}(y|x)}{P_{\rm out}^A(y)}\right]. \tag{17}$$

In the quantum measurement process described by the reversible operation, since the output probability does not depend on the initial quantum state of the physical system [5, 6], the information gain by the quantum measurement becomes zero. This fact may be written as $I(Y_{\rm out}^A; X_{\rm in}^S) = 0$.

To obtain the entropy decrease of the physical system that is caused by the quantum measurement process, we calculate the joint probability $P_{\rm out}^{SA}(x, y)$ that the measurement outcome $y$ is obtained and the intrinsic observable takes the value $x$ in the quantum state of the physical system after the measurement. The joint probability is obtained from equation (11),

$$\begin{aligned} P_{\rm out}^{SA}(x, y) &= P_{\rm out}^S(x|y)P_{\rm out}^A(y) \\ &= {\rm Tr}_S[\hat{\mathcal{X}}_S(x)\hat{\rho}_{\rm out}^S(y)]P_{\rm out}^A(y) \\ &= {\rm Tr}_S[\hat{\mathcal{X}}_S(x)\hat{\mathcal{L}}_S(y)\hat{\rho}_{\rm in}^S] \\ &= {\rm Tr}_S[\hat{\mathcal{L}}_S^\dagger(y)\hat{\mathcal{X}}_S(x)\cdot\hat{\rho}_{\rm in}^S] \\ &= P_{SA}(y|f_y(x))P_{\rm in}^S(f_y(x)). \end{aligned} \tag{18}$$

Then, assuming equation (15), we obtain the joint entropy $H(X_{\rm out}^S, Y_{\rm out}^A)$ in the quantum measurement process described by the partially reversible operation,

$$\begin{aligned} H(X_{\rm out}^S, Y_{\rm out}^A) &= -\sum_{x\in\Omega_X}\sum_{y\in\Omega_Y} P_{\rm out}^{SA}(x, y)\log P_{\rm out}^{SA}(x, y) \\ &= -\sum_{x\in\Omega_X}\sum_{y\in\Omega_Y} P_{SA}(y|x)P_{\rm in}^S(x)\log[P_{SA}(y|x)P_{\rm in}^S(x)] \\ &= H(X_{\rm in}^S) + H(Y_{\rm out}^A) - I(Y_{\rm out}^A; X_{\rm in}^S) \end{aligned} \tag{19}$$

where $H(X_{\rm in}^S)$ and $H(Y_{\rm out}^A)$ are the Shannon entropy calculated by the initial probability $P_{\rm in}^S(x)$ of the physical system and the output probability $P_{\rm out}^A(y)$ of the measurement apparatus, and $I(Y_{\rm out}^A; X_{\rm in}^S)$ is given by equation (17). The conditional entropy $H(X_{\rm out}^S|Y_{\rm out}^A)$ of the physical system after obtaining the measurement outcome becomes

$$\begin{aligned} H(X_{\rm out}^S|Y_{\rm out}^A) &= -\sum_{x\in\Omega_X}\sum_{y\in\Omega_Y} P_{\rm out}^S(x|y)P_{\rm out}^A(y)\log P_{\rm out}^S(x|y) \\ &= H(X_{\rm out}^S, Y_{\rm out}^A) - H(Y_{\rm out}^A) \\ &= H(X_{\rm in}^S) - I(Y_{\rm out}^A; X_{\rm in}^S). \end{aligned} \tag{20}$$

Therefore when we obtain the measurement outcome, the decrease of the entropy of the physical system is given by

$$\Delta H(X_{\rm out}^S, X_{\rm in}^S|Y_{\rm out}^A) = H(X_{\rm in}^S) - H(X_{\rm out}^S|Y_{\rm out}^A) = I(Y_{\rm out}^A; X_{\rm in}^S). \tag{21}$$

This result indicates that the information gain $I(Y_{\rm out}^A; X_{\rm in}^S)$ is equal to the entropy decrease $\Delta H(X_{\rm out}^S, X_{\rm in}^S|Y_{\rm out}^A)$ of the physical system in the quantum measurement process described by the partially reversible operation. Several examples of the quantum measurement process

**Table 1.** Comparison between the reversible operation and the partially reversible operation in quantum measurement processes.

| | Reversible operation | Partially reversible operation |
|---|---|---|
| The reversibility of the quantum operation | Reversible for $\hat{\rho}^S_{\rm in}$ $\hat{\mathcal{R}}_S(y)[\hat{\mathcal{L}}_S(y)\hat{\rho}^S_{\rm in}] \propto \hat{\rho}^S_{\rm in}$ | Reversible for $\hat{\mathcal{X}}_S(x)$ $\hat{\mathcal{R}}^\dagger_S(y)[\hat{\mathcal{L}}^\dagger_S(y)\hat{\mathcal{X}}_S(x)] \propto \hat{\mathcal{X}}_S(x)$ |
| The coefficient in the reversibility relation | The output probability $P^A_{\rm out}(y)$ | The conditional probability $P_{SA}(y\|x)$ |
| The information gain by the measurement | No information gain $I(Y^A_{\rm out}; X^S_{\rm in}) = 0$ | The decrease of the entropy $I(Y^A_{\rm out}; X^S_{\rm in}) = \Delta H(X^S_{\rm out}; X^S_{\rm in}\|Y^A_{\rm out})$ |

described by the partially reversible operation, in which equation (21) holds, are given in [12, 13].

The comparison between the reversible operation and the partially reversible operation is summarized in table 1. Although in this letter, we have confined ourselves to considering quantum measurement processes of discrete observables, the results can be generalized for quantum measurement processes of continuous observables, using the formulation given in [3, 4]. In any quantum measurement process described by a partially reversible operation with respect to the intrinsic observable, which satisfies equation (15), the amount of information about the intrinsic observable is equal to the decrease of the entropy of the physical system caused by the quantum measurement process. The dual map of the partially reversible operation becomes reversible for the intrinsic observable of the physical system. On the other hand, quantum measurement processes described by reversible operations give us no information about the quantum state of the physical system, which is consistent with the indistinguishability of quantum states.

Finally, we consider quantum measurement processes of photon number, which are carried out respectively by means of a lossless beam splitter (BS) [17, 18], nondegenerate parametric amplifier (PA) [17] and degenerate four-wave mixer (FM) [17, 19]. They are described by the partially reversible operations with respect to the photon number. From equation (2), the quantum operation $\hat{\mathcal{L}}_S(y)$ is determined by the POVM $\hat{\mathcal{Y}}_A(y)$ and the initial quantum state $\hat{\rho}^A_{\rm in}$ of the measurement apparatus and the unitary operator $\hat{\mathcal{U}}_{SA}$ which describes the state change by the system–apparatus interaction. The measurement apparatus before the interaction with the physical system is prepared in the vacuum state $\hat{\rho}^A_{\rm in} = |0_A\rangle\langle 0_A|$. Since the photodetection is assumed to be ideal, the POVM of the measurement apparatus is a projection operator $\hat{\mathcal{Y}}_A(m) = |m_A\rangle\langle m_A|$. The intrinsic observable of the physical system is given by a projection operator $\hat{\mathcal{X}}_S(n) = |n_S\rangle\langle n_S|$. Here, $|n_S\rangle$ and $|m_A\rangle$ are the Fock states of the physical system and the measurement apparatus. In the photon-number measurement, $\Omega_X$ and $\Omega_Y$ are the sets of non-negative integers. The unitary operators are given respectively by

$$\hat{\mathcal{U}}^{\rm BS}_{SA} = \exp[-\theta(\hat{a}^\dagger_S\hat{a}_A - \hat{a}_S\hat{a}^\dagger_A)] \tag{22}$$

$$\hat{\mathcal{U}}^{\rm PA}_{SA} = \exp[\theta(\hat{a}^\dagger_S\hat{a}^\dagger_A - \hat{a}_S\hat{a}_A)] \tag{23}$$

$$\hat{\mathcal{U}}^{\rm FM}_{SA} = \exp[-{\rm i}\theta\hat{a}^\dagger_S\hat{a}_S(\hat{a}^\dagger_A + \hat{a}_A)] \tag{24}$$

where $\hat{a}_S$ and $\hat{a}^\dagger_S$ ($\hat{a}_A$ and $\hat{a}^\dagger_A$) are the bosonic annihilation and creation operators of the physical system (the measurement apparatus). The information-theoretical properties of these quantum measurement processes have been investigated in detail [12]. The quantum operation $\hat{\mathcal{L}}_S(m)$ and the reversal operation $\hat{\mathcal{R}}_S(m)$ for these photon-number measurement processes can be expressed as

$$\hat{\mathcal{L}}_S(m)\hat{O}_S = \hat{M}_S(m)\hat{O}_S\hat{M}_S^\dagger(m) \qquad \hat{\mathcal{R}}_S(m)\hat{O}_S = \hat{N}_S(m)\hat{O}_S\hat{N}_S^\dagger(m) \tag{25}$$

which are called the pure quantum operation [5, 6], where the set $\mathcal{M}$ in equation (3) has only one element. After straightforward calculation, we obtain

$$\hat{M}_S^{\mathrm{BS}}(m) = \frac{1}{\sqrt{m!}}\left(\frac{\mathcal{R}}{\mathcal{T}}\right)^{\frac{1}{2}m}\hat{a}_s^m\mathcal{T}^{\frac{1}{2}\hat{a}_S^\dagger\hat{a}_S} \qquad \hat{N}_S^{\mathrm{BS}}(m) = (\hat{E}_S^m)^\dagger \tag{26}$$

$$\hat{M}_S^{\mathrm{PA}}(m) = \frac{1}{\sqrt{m!}}\mathcal{F}^{\frac{1}{2}m}\hat{a}_S^{\dagger m}\mathcal{G}^{\frac{1}{2}\hat{a}_S\hat{a}_S^\dagger} \qquad \hat{N}_S^{\mathrm{PA}}(m) = \hat{E}_S^m \tag{27}$$

$$\hat{M}_S^{\mathrm{FM}}(m) = \frac{1}{\sqrt{m!}}(\theta\hat{a}_S^\dagger\hat{a}_S)^m \mathrm{e}^{-\frac{1}{2}(\theta\hat{a}_S^\dagger\hat{a}_S)^2} \qquad \hat{N}_S^{\mathrm{FM}}(m) = \hat{1}_S \tag{28}$$

where $\hat{E}_S = (\hat{a}_S\hat{a}_S^\dagger)^{-\frac{1}{2}}\hat{a}_S$ is the Susskind–Glogower phase operator of the physical system [20, 21] and $\mathcal{T} = 1 - \mathcal{R} = \cos^2\theta$ and $\mathcal{F} = 1 - \mathcal{G} = \tanh^2\theta$. Since the operator $\hat{E}_S$ is not unitary, but isometric, the quantum operations $\hat{\mathcal{L}}_S^{\mathrm{BS}}(m)$ and $\hat{\mathcal{L}}_S^{\mathrm{PA}}(m)$ are partially reversible by the isometric quantum operations. On the other hand, since the dual map of the quantum operation $\hat{\mathcal{L}}_S^{\mathrm{FM}}(m)$ makes the intrinsic observable $\hat{\mathcal{X}}_S(n)$ unchanged, the reversal operation $\hat{\mathcal{R}}_S^{\mathrm{FM}}(m)$ becomes an identity operation. The conditional probability $P_{SA}(m|n)$ and the function $f_m(n)$ are given respectively by

$$P_{SA}^{\mathrm{BS}}(m|n) = \frac{n!}{m!(n-m)!}\mathcal{R}^m\mathcal{T}^{n-m} \qquad f_m^{\mathrm{BS}}(n) = n + m \tag{29}$$

$$P_{SA}^{\mathrm{PA}}(m|n) = \frac{(n+m)!}{m!n!}\mathcal{F}^m\mathcal{G}^{n+1} \qquad f_m^{\mathrm{PA}}(n) = n - m \tag{30}$$

$$P_{SA}^{\mathrm{FM}}(m|n) = \frac{1}{m!}(\theta n)^{2m}\exp[-(\theta n)^2] \qquad f_m^{\mathrm{FM}}(n) = n \tag{31}$$

which satisfy equation (15). Therefore, the amount of information about the photon number of the physical system obtained by these quantum measurements is equal to the decrease of the Shannon entropy of the physical system [12].

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